Identification of Externally Positive Systems

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***Abstract*— We consider identification of externally positive linear discrete-time systems from input/output data. The proposed method is formulated as a semidefinite program, and is guaranteed to identify models that are ellipsoidal cone-invariant and, consequently, externally positive. We demonstrate empirically that this cone-invariance approach can significantly reduce the conservatism associated with methods that enforce internal positivity as a sufficient condition for external positivity.**

I. Introduction

Systems for which physical constraints imply nonnegativity of the quantities of interest are ubiquitous in applications, being found in areas such as economics, chemistry, medicine, data and electricity networks [9], [11], [19], [23]. For example, consider the linear discrete-time invariant system

$$G: \begin{cases} x_{t+1} = Ax_t + Bu_t, \\ y_t = Cx_t + Du_t, \end{cases} \tag{1}$$

with state vector $x \in \mathbb{R}^{n_x}$, input $u \in \mathbb{R}^{n_u}$ and output $y \in \mathbb{R}^{n_y}$. Here y could be the pH value of a solution within a tank, which is controlled via the flow u of some acid solution. Clearly, u and y are only measured in nonnegative quantities. Systems for which positive inputs lead to positive outputs are said to be *externally positive*. If in addition the state is confined to be nonnegative, the system is referred to as *internally positive*. In modeling applications, it is often important (or even essential) to respect such positivity properties; simulations that fail to do so may lead to questionable conclusions that lack interpretability.

Despite the prevalence of (internally and externally) positive systems, data-driven modeling (a.k.a. identification) of such systems has received little attention. Most of the research effort has focused on the so-called positive realization problem, i.e., determining the conditions for which there exists an internally positive realization of a system with nonnegative impulse response, c.f. [3]. Among the few published results concerned with the identification problem are [4], which presents conditions for 'compartmentality' of identified models; [22], which considers third order internally positive systems with Poisson output; and [14], [25], both concerned with model stability of internally positive systems.

In this paper we suggest convex modifications to the subspace identification method, c.f., [21], [27], which are guaranteed to produce externally positive models. Since in general there is no polynomial time test for external positivity [6], our main idea is to utilize a recently proposed sufficient condition based on ellipsoidal (second-order) cones [13]. This has the advantage of applicability to systems known to have no internally positive realization. Furthermore, we will see that exploiting prior knowledge of external positivity can even improve fidelity of the identified model.

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II. Background

A. Notation

The following notation for real matrices and vectors $X = (x_{ij})$ are used throughout this paper. We say that $X \in \mathbb{R}^{m\times n}_{\geq 0}$ is *non-negative*, if all entries are non-negative ($x_{ij} \geq 0$ for all i, j). We use the notation $X(:, i)$ and $X(j, :)$ to denote the i-th column and j-th row of X.

If $X = X'$, then we write $X \succ 0$, or $X \succeq 0$ if X is positive definite, or semi-definite, i.e. the set of eigenvalues of X, $\sigma(X) \subset [0, \infty]$. The cone of positive definite (semidefinite) $n\times n$ matrices is denoted $\mathbb{S}^n_{++}$ ($\mathbb{S}^n_{+}$). We also use these notations to describe the relation between two matrices, e.g. $A \succeq B$ defines $A - B \succeq 0$. The set of $n \times n$ *skew-symmetric* matrices is denoted $\mathcal{A}_n$, i.e., $X \in \mathcal{A}_n$ implies $X = -X'$.

A real vector valued function $u(t) \in \mathbb{R}^m$ is called *non-negative* if and only if $u(t) \in \mathbb{R}^m_{\geq 0}$ for all $t \geq 0$. The inertia (p, z, n) of X is defined by the number of eigenvalues of X with positive, zero and negative real-parts, respectively counting multiplicities.

The normal distribution with mean μ and covariance Σ is denoted $\mathcal{N}(\mu, \Sigma)$.

B. Cone invariance

Definition 1. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a cone and $A \in \mathbb{R}^{n\times n}$. $\mathcal{K}$ is called A-invariant if and only if $A\mathcal{K} \subset \mathcal{K}$.*

Definition 2. *(A, B) is called cone-invariant w.r.t. a cone $\mathcal{K}$ if and only if $\mathcal{K}$ is A-invariant and $B(:, i) \in \mathcal{K}$ for all i.*

Definition 3 (Ellipsoidal cones)**.** *Let $Q = Q' \in \mathbb{R}^{n\times n}$ with inertia $(n - 1, 0, 1)$, then*

$$\mathcal{K}_Q := \{x : x'Qx \leq 0\}$$

is called an ellipsoidal double-cone. If $p \in \mathbb{R}^n$ is such that

$$\{p\}^\perp \cap \mathcal{K}_Q = \{0\}$$

where $\{p\}^\perp$ denotes the orthogonal complement of linear span $\{p\}$ of p, then $\mathcal{K}_{Q,p} := \{x : x'Qx \leq 0, p'x \geq 0\}$ is called an ellipsoidal cone.

The following is an straight-forward modification of [13, Theorem 1] to A-invariant cones, using the results in [24].

Proposition 1. *Let $Q = Q'$ have inertia $(n-1, 0, 1)$. Then $\mathcal{K}_{Q,p}$ is A-invariant if and only if*

$$\exists \gamma, \tau : A^T Q A - \gamma Q \preceq 0, \quad Q + \tau p p' \succ 0. \tag{2}$$

The cone defined by $\mathcal{L}_+^n := \mathcal{K}_{J_n, e_1}$, where $J_n := \mathrm{diag}(-1, 1, \ldots, 1) \in \mathbb{R}^{n \times n}$ and e_1 denotes the first canonical vector, is commonly referred to as the *Lorentz* cone or sometimes the *ice-cream cone*.

Note that if $\mathcal{K} \subset \mathbb{R}^n$ is a convex cone, then $\{A \in \mathbb{R}^{n \times n} : A\mathcal{K} \subset \mathcal{K}\}$ defines a convex cone. In case of $\mathcal{L}_+^n$, it has been shown in [15] that this cone has a linear matrix inequality (LMI) representation. In order to state these LMIs, let the linear map $W_n : \mathbb{R}^n \to \mathbb{R}^{n-1 \times n-1}$ be defined as

$$W_n(x) := \begin{pmatrix} x_1 + x_2 & x_3 & \cdots & x_n \\ x_3 & x_1 - x_2 & & 0 \\ \vdots & & \ddots & \\ x_n & 0 & & x_1 - x_2 \end{pmatrix}.$$

Then for any $A \in \mathbb{R}^{n \times n}$ with arbitrary rank-1 decomposition $A = \sum_i u_i v_i'$, it is defined that

$$(W_n \otimes W_n)(A) := \sum_i (W_n(u_i) \otimes W_n(v_i)),$$

where $\otimes$ denotes the *Kronecker product* for matrices. Furthermore, let us define

$$\mathcal{A}_{n-1} \otimes \mathcal{A}_{n-1} := \{X \otimes Y : X, Y \in \mathcal{A}_{n-1}\}.$$

Proposition 2. *Let $n \geq 3$ and $A \in \mathbb{R}^{n \times n}$. Then $\mathcal{L}_+^n$ is A-invariant if and only if $\exists X \in \mathcal{A}_{n-1} \otimes \mathcal{A}_{n-1}$ such that*

$$\mathcal{W}_n(A) := (W_n \otimes W_n)(A) + X \succeq 0. \tag{3}$$

In [16] it has been shown that Proposition 2 remains true if $\mathcal{A}_{n-1} \otimes \mathcal{A}_{n-1}$ is replaced by the linear subspace

$$\left\{ M = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix} : M_{ij} \in \mathcal{A} \right\},$$

which is a condition that is easier to implement in conventional software (see [12], [20]).

C. Positive Systems

Next we recall the notations and properties of externally and internally positive systems.

Definition 4. *A linear system* (1) *is called* externally positive *if and only if its output corresponding to a zero initial state is non-negative for every non-negative input.*

Proposition 3. *A linear system (A, B, C, D) is externally positive if and only if $\forall t \geq 0 : Ce^{At}B \in \mathbb{R}_{\geq 0}^{n_y \times n_u}$ and $D \in \mathbb{R}_{\geq 0}^{n_y \times n_u}$. [11]*

It is readily seen that every single-input-single-output (SISO) externally positive system (A, B, C) is invariant with respect to its so-called reachable cone $R(A, B) := \mathrm{cl}(\mathrm{cone}\{A^k B : k \in \mathbb{N}_0\})$, where $\mathrm{cl}(\cdot)$ denotes the topological closure and $\mathrm{cone}(\cdot)$ the *convex conic hull*. In fact, this is the smallest A-invariant cone that includes B, i.e. if (A, B) is invariant w.r.t. to a cone $\mathcal{K}$, then $R(A, B) \subset \mathcal{K}$.

In general, it is NP-hard to test whether a system is externally positive. However, in [13] the following tractable test has been suggest. Assume that A has a simple dominant pole, then (A, B) will always be invariant w.r.t an ellipsoidal cone $\mathcal{K}_Q$. Hence, letting the dual cone of $\mathcal{K}_Q$ be defined as

$$\mathcal{K}_Q^* := \{y \in \mathbb{R}^n : x'y \geq 0 \text{ for all } x \in \mathcal{K}_Q\},$$

external positivity can be concluded if all columns of C lie within $\mathcal{K}_Q^*$ and $D \in \mathbb{R}_{\geq 0}^{n_y \times n_u}$. This motivates the following definition, which implies external positivity.

Definition 5. *A linear system (1) is called cone-positive w.r.t. a cone $\mathcal{K}$ if and only if $\mathcal{K}$ is A-invariant, $B(:, i) \in \mathcal{K}$ and $C(j, :)' \in \mathcal{K}^*$ for all i and j.*

Definition 6. *A linear system* (1) *is called internally positive if and only if its state and output are non-negative for every non-negative input and every non-negative initial state.*

Internal positivity of (1) requires that the non-negative orthant $\mathbb{R}_{\geq 0}^{n_x}$ is invariant w.r.t. A. In [5] it is shown, that this is the case if and only if $A \in \mathbb{R}_{\geq 0}^{n_x \times n_x}$.

Proposition 4. *A discrete linear system (A, B, C, D) is internally positive if and only if A, B, C, D are non-negative [11].*

Note that if a system admits an internally positive realization, then (A, B) must be invariant w.r.t to a polyhedral cone. In contrast to ellipsoidal cones, there are examples of externally positive systems where this is not the case [3].

D. Problem statement

The following problem is investigated in this paper: Given measured input-output data $\{\tilde{u}_t, \tilde{y}_t\}_{t=1}^N$ from an externally positive system, find a cone-positive system. Since the simultaneous search for system matrices and the cone is nonconvex, we employ the following two step strategy: (i) estimate internal states from the input/output data and find a cone that encloses these states; (ii) find a system that is cone positive w.r.t to the estimated cone and minimize a convex measure of model fidelity (namely, *equation error*, c.f. (7a)).

III. Subspace Identification of Externally Positive Systems

A. Subspace identification

Since their introduction in the 1990s [18], subspace methods have become an indispensable tool for the identification of linear dynamical systems; c.f. [21] for a recent survey. For the purpose of this paper, a subspace identification algorithm SS can be interpreted as a singular value decomposition (SVD) of a weighted matrix, constructed from measured input/output data $\{\tilde{u}, \tilde{y}\}$, that yields an estimate $\tilde{x}$ of the internal states of a linear system, i.e.

$$\{\tilde{x}\}_{t=1}^N = SS(\{\tilde{u}, \tilde{y}\}_{t=1}^N).$$

For a purely deterministic system,[1] i.e.,

$$x_{t+1}^d = Ax_t^d + B\tilde{u}_t, \quad \tilde{y} = Cx_t^d + D\tilde{u}_t \tag{4a}$$

[1] In the stochastic case, where additive Gaussain noise effects the states and measurements in (4), the subspace algorithm returns $\tilde{x}$ corresponding to the Kalman filter state estimates, c.f. [27, §4.2].

the state estimate is exact, up to a similarity transformation, i.e. $\tilde{x}_t = M_{\text{ss}} x_t^d$, c.f. equation 2.13 in [27, §2.2]. The user selected weights on the data matrix influence the transformation M_{ss}, and give rise to the different subspace algorithms, e.g. N4SID, CVA, MOESP, c.f. [27, §4.3]. For identification of generic LTI systems, the unknown transformation M_{ss} influences the state-space realization of the identified system, but has no effect on the input-output dynamics.

For identification of internally positive systems, the arbitrary state transformation M_{ss} is problematic, as there is no guarantee that the state estimates $\tilde{x}$ will be consistent with an internally positive realization of the dynamics. Specifically, $M_{\text{ss}} A M_{\text{ss}}^{-1}$, $M_{\text{ss}} B$, or $C M_{\text{ss}}^{-1}$ may not be nonnegative, even if (4) represents an internally positive system. This is a consequence of the fact that internal positivity is not preserved under arbitrary similarity transformations. In such cases, subspace identification subject to internal positivity constraints (i.e. nonnegativity of the system matrices) as in [17], [25] may lead to poor performance, even in the absence of noise. In contrast, cone-positivity is just an input-output property, which is preserved under arbitrary similarity transformations, and thus M_{ss} provides no fundamental barrier to identification of such systems.

For clarity of exposition in the sequel, let us define the following convex parameterization of models that are cone-positive w.r.t. $\mathcal{L}_+^{n_x}$,

$$\Theta_{\text{CP}} := \{A, B, C, D : \mathcal{W}_{n_x}(A) \succeq 0, \tag{5a}$$

$$B(:,i) \in \mathcal{L}_+^{n_x},\ i = 1, \ldots, n_u \tag{5b}$$

$$C(i,:) \in \mathcal{L}_+^{n_x},\ i = 1, \ldots, n_y \tag{5c}$$

$$D \in \mathbb{R}_{\geq 0}^{k \times m}\}. \tag{5d}$$

B. Basic approach

The properties discussed in Section III-A motivate the following simple procedure for identifying systems invariant w.r.t. $\mathcal{L}_+^{n_x}$.

1. Given nonnegative $\{\tilde{u}_t, \tilde{y}_t\}_{t=1}^N$, obtain state estimates $\{\tilde{x}_t\}_{t=1}^N$ using a subspace algorithm, e.g. N4SID [26].
2. Search for an ellipsoidal cone $\mathcal{K}_{\tilde{Q}}$ that encloses $\{\tilde{x}_t\}_{t=1}^N$. Let $P^* \in \mathbb{S}^{n_x}$ and $p^* \in \mathbb{R}^{n_x}$ denote the optimal solutions to the convex program

$$\min_{P,p} \quad -\log\det P + \|p\| \tag{6a}$$

$$\text{s.t.} \quad p'\tilde{x}_t \geq \|P\tilde{x}_t\|,\ t = 1, \ldots, T \tag{6b}$$

$$P \succeq I. \tag{6c}$$

   Then $\tilde{Q} = P^{*'}P^* - p^* p^{*'}$ defines a suitable cone $\mathcal{K}_{\tilde{Q}}$. This approach is motivated by the *Löwner-John* ellipsoid (see [8]).
3. Let $\tilde{Q} = U\Lambda U'$ denote the eigendecomposition of $\tilde{Q}$. Then applying the transformation $\tilde{T} = U|\Lambda|^{\frac{1}{2}}$ ensures that $\bar{x} := \tilde{T}\tilde{x}_t \in \mathcal{L}_+^{n_x},\ \forall t$.
4. Minimize *least squares equation error* subject to cone-invariance constraints, i.e. solve the convex program

$$\min_{A,B,C,D} \quad \mathcal{E}(\bar{x}) := \sum_{t=1}^{N-1} |\epsilon_t(\bar{x})|^2 + \sum_{t=1}^{N} |\eta_t(\bar{x})|^2 \tag{7a}$$

$$\text{s.t.} \quad (A, B, C, D) \in \Theta_{\text{CP}} \tag{7b}$$

   where ϵ and η denote the *equation errors*

$$\epsilon_t(z) = z_{t+1} - Az_t - B\tilde{u}_t, \quad \eta_t(z) = \tilde{y}_t - Cz_t - D\tilde{u}_t.$$

We note, in passing, that equation error $\mathcal{E}$, as in (7a), is the usual cost function minimized in subspace identification.

The method outlined above can be solved as a semidefinite program [28], for which many good general-purpose solvers exist, and guarantees cone-invariance of the identified system w.r.t. $\mathcal{L}_+^{n_x}$. However, the effectiveness of this approach depends on our ability to accurately estimate the cone $\mathcal{K}_{\tilde{Q}}$. Unfortunately, it is unlikely that our estimate $\mathcal{K}_{\tilde{Q}}$ will coincide perfectly with cone associated with the optimal cone-positive system, for the following reasons:

i. The measurements are usually corrupted by noise and so state estimates are not perfect.
ii. Nonnegative inputs only permit the internal states to occupy a subset of the reachable cone, which is not necessarily an ellipsoidal cone.
iii. The closer the states are to the boundary of the reachable cone, the more accurately the cone can be estimated. However, it is unknown how to generate such an input sequence.

C. Cone estimation given approximate system

Given the difficulties associated with estimating the cone $\mathcal{K}_{\tilde{Q}}$ from the states $\tilde{x}$ alone, we propose an alternative method in which we first identify a model of the system using standard subspace techniques, and then use this model to estimate the cone. Specifically, after performing Steps 1-3 outlined in Section III-B, we solve the convex program

$$(A_{\text{ls}}, B_{\text{ls}}, C_{\text{ls}}, D_{\text{ls}}) = \arg\min_{A,B,C,D} \sum_{t=1}^{N-1} |\epsilon_t(\bar{x})|^2 + \sum_{t=1}^{N} |\eta_t(\bar{x})|^2. \tag{8}$$

This unconstrained minimization can be accomplished by linear least squares, as in standard subspace identification methods. We now use the system $(A_{\text{ls}}, B_{\text{ls}}, C_{\text{ls}}, D_{\text{ls}})$ to search for a cone $\mathcal{K}_{\hat{Q}}$, where $\hat{Q}$ is parameterized by

$$\hat{Q} = \begin{bmatrix} -r_0 & r_1' \\ r_1 & R \end{bmatrix} \in \mathbb{S}^{n_x}$$

for $r_0 \in \mathbb{R}_{>0}$, $r_1 \in \mathbb{R}^{n_x - 1}$ and $R \in \mathbb{S}_{++}^{n_x - 1}$. Notice that $\hat{Q}$ has inertia $(n_x, 0, -1)$ by construction. To identify $\hat{Q}$ we solve the program

$$\begin{aligned} \min_{\delta_1,\delta_2,\delta_3,\hat{Q},\gamma,\tau} \quad & \delta_1 + \delta_2 + \delta_3 \\ \text{s.t.} \quad & A_{\text{ls}}'\hat{Q}A_{\text{ls}} - \gamma\hat{Q} \preceq \delta_1 I \\ & B_{\text{ls}}(:,i)'\hat{Q}B_{\text{ls}}(:,i) \leq \delta_2, \quad i = 1, \ldots, n_u \\ & \hat{Q} + \tau C_{\text{ls}}(i,:)C_{\text{ls}}(i,:)' \succeq -\delta_3 I, \quad i = 1, \ldots, n_y \\ & \delta_1, \delta_2, \delta_3, \tau \geq 0 \end{aligned} \tag{9}$$

where $\delta_1, \delta_2, \delta_3, \tau, \gamma \in \mathbb{R}$. Note that (9) is convex (SDP) for constant γ. In practice, a coarse grid-search over $\gamma \in [0, 1]$ is sufficient to solve (9) and obtain $\hat{Q}$.

When the true system generating the problem data is ellipsoidal cone-positive, and in the absence of any measurement noise, $(A_{\text{ls}}, B_{\text{ls}}, C_{\text{ls}}, D_{\text{ls}})$ will also be cone-positive w.r.t. some ellipsoidal cone, which will be recovered by (9), with $\delta_1 = \delta_2 = \delta_3 = 0$. In the presence of measurement noise, (8) will not generally return an ellipsoidal cone-positive system. Nevertheless, we demonstrate empirically that in such situations the above procedure yields useful cone estimates, $\mathcal{K}_{\hat{Q}}$, c.f. Section IV-B.

Finally, with the cone $\mathcal{K}_{\hat{Q}}$ returned by (9), we apply the transformation $\hat{T} = \hat{U}|\hat{\Lambda}|^{\frac{1}{2}}$ to obtain $\hat{x}_t := \hat{T}\bar{x}_t = \hat{T}\tilde{T}\tilde{x}_t$, where $\hat{Q} = \hat{U}\hat{\Lambda}\hat{\Lambda}'$ denotes the eigendecomposition of $\hat{Q}$. With these transformed states, we minimize $\mathcal{E}(\hat{x})$ subject to cone-invariance constraints, i.e. we solve

$$\min_{\hat{A},\hat{B},\hat{C},\hat{D}} \quad \mathcal{E}(\hat{x}) = \sum_{t=1}^{N-1} |\epsilon_t(\hat{x})|^2 + \sum_{t=1}^{N} |\eta_t(\hat{x})|^2 \tag{10a}$$

$$\text{s.t.} \quad (\hat{A},\hat{B},\hat{C},\hat{D}) \in \Theta_{\text{CP}}. \tag{10b}$$

Remark 1. *The performance of the methods outlined in both Section III-B and Section III-C can sometimes be improved by grid searching over a one-dimensional scaling parameter, $\alpha > 0$, such that we minimize $\mathcal{E}(\bar{x}^\alpha)$ and $\mathcal{E}(\hat{x}^\alpha)$ in (7) and (10), respectively, where $\bar{x}^\alpha = \text{diag}(\alpha, 1, \dots)\bar{x}$ and $\hat{x}^\alpha = \text{diag}(\alpha, 1, \dots)\hat{x}$. This scaling of the states can help compensate for inaccuracies in the estimated cone, that arise due to the reasons outlined in Section III-B.*

D. State transformations for internal positivity constraints

Despite the fact that there are systems which do not leave a polyhedral cone invariant, our basic approach can also be used by replacing $\mathcal{K}_{\tilde{Q}}$ with a polyhedral cone. This has the advantage that also an internally positive realization can be obtained (see [2], [10]). The purpose of this section is to provide a method of identifying systems subject to internal positivity constraints, for comparison against our proposed cone-invariance method in the sequel.

The easiest way to obtain a polyhedral cone that encloses $\{\tilde{x}_t\}_{t=1}^N$ is to use $\text{cone}(\{\tilde{x}_t\}_{t=1}^N)$. Then instead of applying a state transformation, we can directly modify the least squares estimation to

$$\min_{A,B,C,D,P_A,P_B} \quad \mathcal{E}(\bar{x}) := \sum_{t=1}^{N-1} |\epsilon_t(\bar{x})|^2 + \sum_{t=1}^{N} |\eta_t(\bar{x})|^2 \tag{11a}$$

$$\text{s.t.} \quad AX = XP_A,\ P_A \in \mathbb{R}_{\geq 0}^{N\times N} \tag{11b}$$

$$B = XP_B,\ P_B \in \mathbb{R}_{\geq 0}^{N\times m} \tag{11c}$$

$$CX \in \mathbb{R}_{\geq 0}^{k\times N}, \tag{11d}$$

$$D \in \mathbb{R}_{\geq 0}^{k\times m}. \tag{11e}$$

where $\epsilon_t(\bar{x})$ and $\eta_t(\bar{x})$ are defined as before.

IV. Case studies

A. Identification of internally positive systems

In this section we apply the proposed method to identification of internally positive systems. Specifically, we compare the following three methods:

i. *Cone-invariant*: the procedure outlined in Section III-C.
ii. *Least squares*: unconstrained minimization of equation error, i.e. (8), as in standard subspace methods.
iii. *Internally positive*: the procedure (11) outlined in Section III-D.

The numerical experiment consists of 300 trials. In each trial, we randomly generate a 3rd order ($n_x = 3$) SISO stable internally positive system, and simulated for $N = 400$ time steps, subject to a nonnegative Gaussian input; i.e. $\tilde{u}_t = |w_t|$ where $w_t \sim \mathcal{N}(0,1)$. The simulated output is then corrupted by additive Gaussian noise, to achieve a signal-to-noise ratio (SNR) between 1-30dB (the SNR is randomly selected in each trial). These inputs and outputs are then passed as problem data to the three methods listed above.

The results of the experiment are depicted in Figure 1, in which we plot the H_∞ error between the identified model and the true model. Figure 1(a) compares the performance of *least squares* to our proposed *cone-invariant* method. We draw two observations from the scatter plot. First, the *cone-invariant* method generally outperforms *least squares*, achieving lower error in 75% of the trials. Secondly, in 11% of trials, the system from *least squares* is not externally positive (such trials are indicated by a red cross in Figure 1(a)). Furthermore, such systems tend to have the worst H_∞ error, which is not surprising, as the true system is internally (and therefore externally) positive.

In Figure 1(b) we compare the *internally positive* method to our proposed *cone-invariant* approach, and observe that the latter achieves lower H_∞ error in the majority (78%) of trials. This illustrates the fundamental difficulty of identifying internally positive systems from input/output data: even after a search for a polyhedral cone, as in (11), it is challenging to find a basis for the states from subspace that is consistent with a positive realization of the dynamics.

B. Identification of externally positive systems

In this section we apply the proposed method to identification of externally positive systems, that are not necessarily internally positive. Specifically, we compare the same three methods as in Section IV-A, in addition to the method:

iv. *Cone-invariant (basic)*: the "basic" procedure outlined in Section III-B.

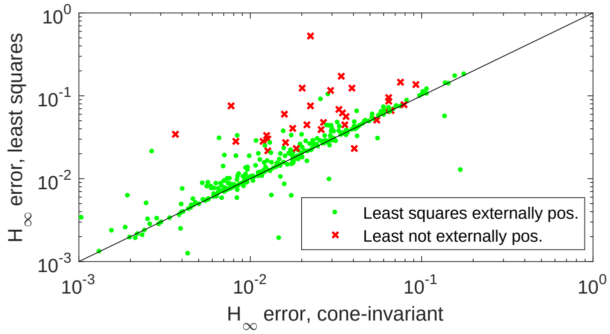
To generate the externally positive systems, we randomly generate

$$A = 0.9 \begin{bmatrix} \cos(\frac{\sqrt{2}}{\pi}) & \sin(\frac{\sqrt{2}}{\pi}) & a_{13} \\ -\sin(\frac{\sqrt{2}}{\pi}) & \cos(\frac{\sqrt{2}}{\pi}) & a_{23} \\ a_{31} & a_{32} & 1 \end{bmatrix}$$

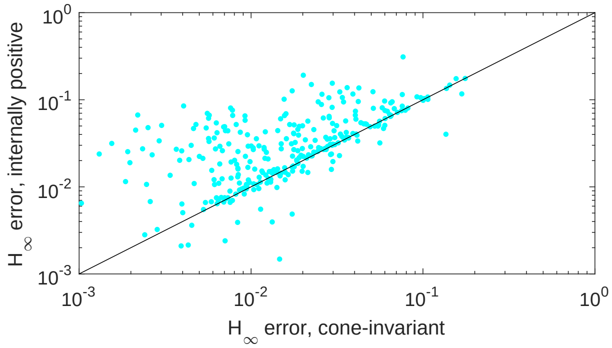
where $a_{13}, a_{23}, a_{31}, a_{32} \sim \mathcal{N}(0, 0.04)$, and set $B = [0.5, 0.6, 1]'$, $C = [0.5, 0.5, 1]$ and $D = 0$. This gives a system similar to the one provided by [3] as an example of a cone-positive system with no internally positive realization.[2].

The numerical experiment consists of 700 trials, following the same procedure described in Section IV-A, except with randomly generated externally positive systems, as described above. The results are depicted in Figure 2. Figure 2(a) compares the performance of our *cone-invariant* method to the more basic *cone-invariant (basic)* approach outlined in Section III-B. The merits of searching for a cone via the

[2]After each random generation, we check for external positivity of the system; only externally positive systems are considered in the experiment.

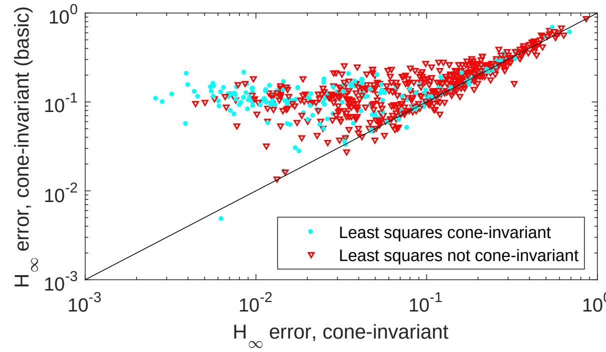


(a) H_∞ error for *cone-invariant* vs *least squares*. Red crosses denote identified systems from *least squares* that are not externally positive, whereas green dots denote systems that are externally positive.

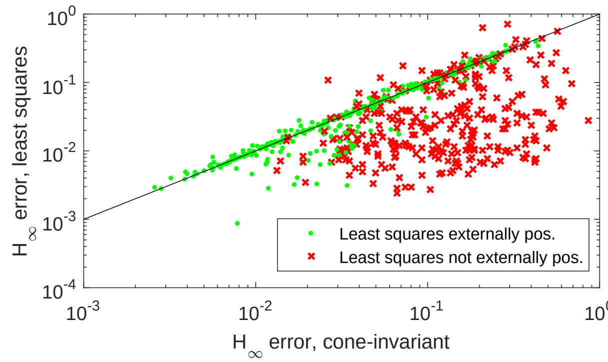


(b) H_∞ error for *cone-invariant* vs *internally positive*.

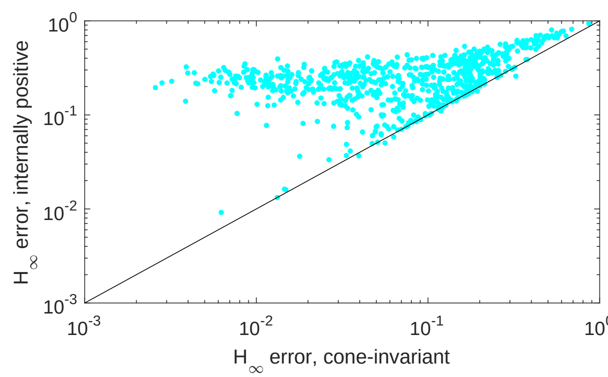
Fig. 1: H_∞ error between identified model and true model, where the true model is internally positive, for 300 different experimental trials; c.f. Section IV-A for details.



(a) H_∞ error for *cone-invariant* vs *cone-invariant (basic)*. The red triangles denote cases in which the model from (8) was *not* cone-positive.



(b) H_∞ error for *cone-invariant* vs *least squares*. Red crosses denote identified systems from *least squares* that are not externally positive, whereas green dots denote systems that are externally positive.



(c) H_∞ error for *cone-invariant* vs *internally positive*.

Fig. 2: H_∞ error between identified model and true model, where the true model is externally positive, for 700 different experimental trials; c.f. Section IV-B for details.

program (9), rather than simply fitting a cone to the states from subspace, is clearly illustrated; the former achieves lower H_∞ error in 84% of the trials. Interestingly, even when the system recovered from (8) was *not* cone-positive (such cases are denoted by a red triangle in Figure 2(a), and imply that (9) cannot be solved with $\delta_1 + \delta_2 + \delta_3 = 0$), the proposed method generally performs much better than the basic approach.

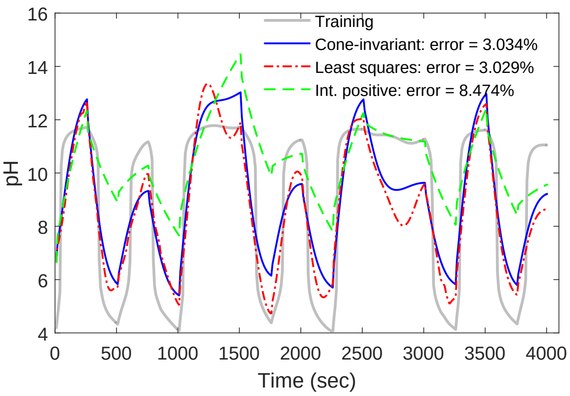
Figure 2(b) compares the performance of our *cone-invariant* method to simple least squares. Unlike identification of internally positive systems, *least squares* generally performs better than the *cone-invariant* method. However, in the majority of these cases, the model from *least-squares* is not externally positive. If we restrict the comparison to externally positive models, then on balance both methods perform comparably. The key point to emphasize here is that the proposed *cone-invariant* method guarantees external positivity, whereas *least squares* fails to return an externally positive model in 45% of trials.

Finally, in Figure 2(c) we compare the *internally positive* method to our proposed *cone-invariant* approach, and observe the latter outperforms the former in 96% of trials. This is not surprising, as the true system is designed to not have an internally positive realization, and demonstrates the conservatism introduced by enforcing internal positivity as a means to ensure external positivity.

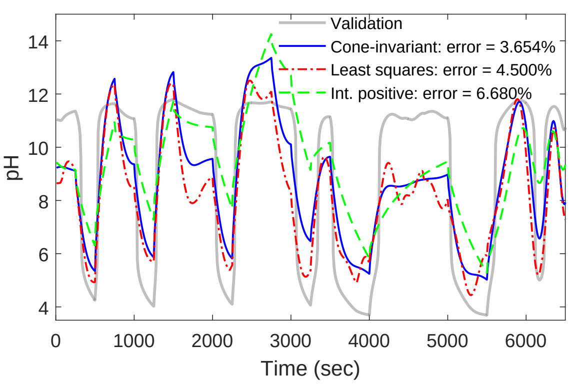
C. pH neutralization in a stirred tank

In this section we identify a model for pH neutralization in a stirred tank. This system has two types of acid flows as the input and the pH value of the tank content as its output. This is a standard benchmark identification problem with data from the *DaISy* database (see [7]). We apply the same three identification algorithms considered in Section IV-A. As with all the case studies, we use N4SID to obtain state estimates. For this problem, the state dimension was $n_x = 3$.

The simulations based on the training and validation data are shown in Fig. 3(a) and Fig. 3(b), respectively. The normalized simulation error, defined $\mathcal{E}_{\text{NSE}} = \frac{\sum_t |\tilde{y}_t - y_t|^2}{\sum_t |\tilde{y}_t|^2}$ where y denotes the simulated output and $\tilde{y}$ denotes measured data (for training or validation), is also reported. We make three remarks about Figure 3. First, the system is highly nonlinear, and so it is difficulty to achieve high fidelity modeling with a linear system. Nevertheless, we observe that our proposed *cone-invariant* method is competitive with standard subspace methods (*least squares*), achieving comparable training error but lower error on validation data. Finally, we note that the model from *least squares* is not externally positive. Our proposed *cone-invariant* method guarantees external positivity with far better performance than methods that impose internal positivity constraints.



(a) Simulated output on training data.



(b) Simulated output on validation data.

Fig. 3: Simulated performance for the pH neutralization process in a stirred tank. Our proposed *cone-invariant* method is compared to *least squares* (as in standard subspace methods), as well the method of Section III-D, which constrains the model to be *internally positive*.

V. Conclusions and future work

We proposed several approaches for system identification with external positivity guarantees. Such models have the advantage of respecting very common physical constraints. In contrast, conventional methods may lead to models that are not externally positive; c.f. Section IV-B. This can be problematic, for instance, if the model is used for simulations. Moreover, as shown in our case studies, enforcing external positivity may even improve the fidelity of the identified model, and generally performs better than enforcing internal positivity as a sufficient condition for external positivity. In the future work, it could be interesting to consider other characterizations of external positivity such as in [1].

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